Math 10A with Professor Stankova
Worksheet, Midterm I Review; Monday, 9/25/2017
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Derivatives

Chain Rule

1. Find the derivative of $f(x) = \cos^3(\frac{1}{3x+1})$.

Solution: We use the chain rule to get that

$$\begin{aligned}\frac{df}{dx}(x) &= 3\cos^2\left(\frac{1}{3x+1}\right)\cdot\frac{d}{dx}\cos\left(\frac{1}{3x+1}\right)\\ &= 3\cos^2\left(\frac{1}{3x+1}\right)\cdot\left[-\sin\left(\frac{1}{3x+1}\right)\cdot\frac{d}{dx}\frac{1}{3x+1}\right]\\ &= 3\cos^2\left(\frac{1}{3x+1}\right)\cdot\left[-\sin\left(\frac{1}{3x+1}\right)\right]\cdot\frac{-1}{(3x+1)^2}\cdot\frac{d}{dx}3x\\ &= 3\cos^2\left(\frac{1}{3x+1}\right)\cdot\sin\left(\frac{1}{3x+1}\right)\cdot\frac{3}{(3x+1)^2}.\end{aligned}$$

2. Find the derivative of $f(x) = x^5\cos(3x)$.

Solution: We use the product rule to get that the derivative is

$$\begin{aligned}x^5\cdot\frac{d}{dx}\cos(3x)+\cos(3x)\cdot\frac{d}{dx}x^5 &= x^5\cdot-\sin(3x)\cdot 3+\cos(3x)\cdot 5x^4.\\ &= -3x^5\sin(3x)+5x^4\cos(3x).\end{aligned}$$

3. Find the derivative of $\sqrt{\sin(2x)}$.

Solution: Using the chain rule, this is

$$\frac{d}{dx}(\sin(2x))^{1/2} = \frac{1}{2}\cdot(\sin(2x))^{-1/2}\cdot\cos(2x)\cdot 2 = \frac{\cos(2x)}{\sqrt{\sin(2x)}}.$$

4. Find the derivative of $\sin(\sqrt{x})$.

Solution: Again we use the chain rule to get

$$\frac{d}{dx}\sin(x^{1/2}) = \cos(x^{1/2}) \cdot \frac{1}{2} \cdot x^{-1/2} = \frac{\cos(\sqrt{x})}{2\sqrt{x}}.$$

5. Find the derivative of $\cot(3x^2)$.

Solution: Chain rule gives

$$\frac{d}{dx}\cot(3x^2) = -\csc^2(3x^2) \cdot 6x = -6x\csc^2(3x^2).$$

Inverse Functions

6. Find the derivative of $\operatorname{arccot}(x)$.

Solution: We know that $\cot(\operatorname{arccot}(x)) = x$ by definition and so taking the derivative with respect to x of both sides of the equation gives

$$-\csc^2(\operatorname{arccot}(x)) \cdot \operatorname{arccot}'(x) = 1,$$

and hence the derivative of $\operatorname{arccot}(x)$ is $-(\csc(\operatorname{arccot}(x)))^{-2}$. We want to get rid of the $\csc(\operatorname{arccot}(x))$ part and get it terms of a polynomial of x. In order to do so, we use the "Pythagorean Theorem" of trigonometry, which is $\sin^2(x) + \cos^2(x) = 1$. Dividing both sides by $\sin^2(x)$ gives $1 + \cot^2(x) = \csc^2(x)$ and hence

$$\frac{-1}{\csc^2(\operatorname{arccot}(x))} = \frac{-1}{1 + [\cot(\operatorname{arccot}(x))]^2} = \frac{-1}{1+x^2}.$$

7. Find the derivative of $\arcsin(x)$.

Solution: Starting with $\sin(\arcsin(x)) = x$, we take the derivative to get

$$\cos(\arcsin(x)) \cdot \arcsin'(x) = 1,$$

and hence the derivative is $\frac{1}{\cos(\arcsin(x))}$. Now we use the fact that $\sin^2(x)+\cos^2(x) = 1$, so that $\cos(x) = \sqrt{1-\sin^2(x)}$, to get that the derivative is

$$\frac{1}{\sqrt{1-\sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1-x^2}}.$$

8. Let $f(x) = xe^x$ and let g be the inverse function of f. Given that $f(1) = e$, find $g'(e)$.

Solution: Since g is the inverse of f, we know that $f(g(x)) = x$ and thus taking the derivative gives $f'(g(x)) \cdot g'(x) = 1$. We plug in $x = e$ to get $f'(g(e)) \cdot g'(e) = 1$. Since $f(1) = e$ and g is the inverse to f, we know that $g(f(1)) = 1$, and hence $g(e) = 1$. Now calculate that $f'(x) = x \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x = xe^x + e^x$. Therefore, we have that

$$g'(e) = \frac{1}{f'(g(e))} = \frac{1}{f'(1)} = \frac{1}{1 \cdot e^1 + e^1} = \frac{1}{2e}.$$

9. Let $f(x) = x^5 + 4x^3$ and let g be the inverse function of f. Given that $f(1) = 5$, find $g'(5)$.

Solution: Since $f(1) = 5$, we know that $g(5)$. Also, note that $f'(x) = 5x^4 + 12x^2$. So, we have that

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{5+12} = \frac{1}{17}.$$

10. Let $f(x) = x^5 + x + 1$ and let g be the inverse function of f. Find the derivative of g at $(3, 1)$.

Solution: Note that $f'(x) = 5x^4 + 1$. We have that

$$g'(x) = \frac{1}{f'(y)} \implies g'(3) = \frac{1}{f'(1)} = \frac{1}{5+1} = \frac{1}{6}.$$

Implicit Differentiation

11. Find the derivative of the function $y^2(6-x) = x^3$ at $(3, 3)$.

Solution: We take the derivative of both sides with respect to x and use implicit differentiation to get that

$$(6-x)\frac{d}{dx}y^2 + y^2 \cdot \frac{d}{dx}(6-x) = \frac{d}{dx}x^3$$
$$\implies (6-x)(2y)\cdot\frac{dy}{dx} + y^2(-1) = 3x^2$$
$$\implies \frac{dy}{dx} = \frac{3x^2+y^2}{2y(6-x)}.$$

We want to find the derivative at the point $(3,3)$, so when $x=y=3$. Plugging that into the equation, we get that value

$$\frac{dy}{dx} = \frac{3(3^2)+3^2}{2(3)(6-3)} = \frac{36}{18} = 2.$$

12. Find $\frac{dy}{dx}$ given that $\frac{1}{y}+\frac{1}{x^2}=1$.

Solution: We take the derivative to get that

$$-y^{-2}\cdot y' + (-2x^{-3}) = 0 \implies y' = \frac{2x^{-3}}{-y^{-2}} = \frac{-2y^2}{x^3}.$$

13. Let $y^2 = x^2(x-1)$. At what points is $\dfrac{dy}{dx}$ not defined?

Solution: We use implicit differentiation to get that

$$2y\frac{dy}{dx} = 3x^2-2x \implies \frac{dy}{dx} = \frac{3x^2-2x}{2y}.$$

Thus, the derivative is not defined whenever $y=0$. So, we need to find all points on the curve such that $y=0$. Plugging that in gives $x^2(x-1)=0$ and hence $x=0,1$ are the solutions. So, the derivative is not defined at the points $(0,0)$ and $(1,0)$.

14. Find $\frac{dy}{dx}$ if $\ln(xy) = e^y$.

Solution: Taking the derivative of both sides gives

$$\frac{1}{xy} \cdot (x \cdot y' + y \cdot 1) = e^y \cdot y'.$$

So, we have that

$$y'e^y - \frac{y'}{y} = \frac{1}{x},$$

and so

$$\frac{dy}{dx} = \frac{1}{x(e^y - 1/y)}.$$

15. Find when the curve $x^4 = 2x^2 - y^2$ has a horizontal derivative.

Solution: Taking the derivative gives $4x^3 = 4x - 2yy'$ and hence

$$\frac{dy}{dx} = \frac{4x - 4x^3}{2y} = \frac{2x - 2x^3}{y}.$$

When the derivative is equal to 0, we have that $2x - 2x^3 = 0$ and hence $2x(1-x^2) = 0$ so $x = 0$ or $x = \pm 1$. Plugging $x = -1$ into the original equation gives $1 = 2 - y^2$ so $y = \pm 1$ and plugging $x = 1$ into the original equation gives $y = \pm 1$ as well. Finally, plugging $x = 0$ into the original equation gives $y = 0$. But, note that at this point, the derivative is not defined since y is in the denominator. Solving for y as $y = \sqrt{2x^2 - x^4}$ gives

$$\frac{dy}{dx} = \frac{2x - 2x^3}{\sqrt{2x^2 - x^4}} = \frac{2x - 2x^3}{x\sqrt{2 - x^2}} = \frac{2 - 2x^2}{\sqrt{2 - x^2}},$$

and by doing this, we see that the slope at $(0, 0)$ is not 0. Thus, there are only 4 points where the derivative is 0, namely $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$.

L'Hopital's Rule

16. Find $\lim_{x \to \infty} \sqrt{2x+1} - \sqrt{x+1}$.

Solution: Plugging in $x = \infty$ gives $\infty - \infty$, which is an indeterminate. We multiply by the conjugate as our first problem solving technique to get

$$\lim_{x \to \infty} \sqrt{2x+1} - \sqrt{x+1} = \lim_{x \to \infty} \sqrt{2x+1} - \sqrt{x+1} \cdot \frac{\sqrt{2x+1} + \sqrt{x+1}}{\sqrt{2x+1} + \sqrt{x+1}}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{2x+1}+\sqrt{x+1}}.$$

Plugging in $x = \infty$ give $\frac{\infty}{\infty}$ which again is an indeterminate. But, this time, we can use L'Hopital's rule and take the derivative of the top and bottom to get that this integral is equal to

$$\lim_{x\to\infty} \frac{1}{(2x+1)^{-1/2}+(2\sqrt{x+1})^{-1}}.$$

Now plugging in $x = \infty$ gives

$$\frac{1}{1/\infty + 1/\infty} = \frac{1}{0^+ + 0^+} = \frac{1}{0^+} = \infty.$$

Thus, the original limit is equal to ∞.

17. Find $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}$.

Solution: Plugging in ∞ gives ∞/∞ which is an indeterminate and hence we try to use L'Hopital's rule. Doing so gives

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\frac{2x}{2\sqrt{x^2-1}}} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x}.$$

Plugging in ∞ and using L'Hopital's rule again gives

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}},$$

so it leads us back to where we originally were. We need another technique to solve this. Since this is an infinite limit with $x \to \pm\infty$, we can divide the top and bottom by the largest power of x that we see, which is just x. Doing so gives

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-1/x^2}} = \frac{1}{\sqrt{1-1/\infty}} = \frac{1}{1} = 1.$$

So, sometimes we need to use other techniques.

18. Find $\lim_{x\to\infty} x^{1/x}$.

Solution: Plugging in ∞ gives ∞^0, which is an indeterminate. In order to get rid of an exponential indeterminate, we can raise everything to the eth power. So

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} (e^{\ln x})^{1/x} = \lim_{x\to\infty} e^{(\ln x)/x}.$$

It suffices to compute the limit of $(\ln x)/x$ as $x \to \infty$. Plugging in ∞ gives ∞/∞ which is an indeterminate and we can use L'Hopital's rule to get that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{1}{\infty} = 0.$$

Thus, the original limit was $e^0 = 1$.

19. Find $\lim_{x\to 0^+} x^x$.

Solution: Plugging in 0^+ gives 0^0, which is an exponential indeterminate. Thus, we have that

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}.$$

So, we need to find the derivative of $x \ln x = \ln x/(1/x)$. Plugging in 0^+ gives the indeterminate $-\infty/\infty$. Thus, we can use L'Hopital's rule to get

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

Thus, the original limit was $e^0 = 1$.

20. Find $\lim_{x\to 0} \dfrac{\cos(x) - 1 + \sin(x)^2/2}{x^4}$.

Solution: We repeatedly use L'Hopital's rule to get

$$\lim_{x\to 0} \frac{\cos(x) - 1 + \sin(x)^2/2}{x^4} = \lim_{x\to 0} \frac{-\sin(x) + 2\sin(x)\cos(x)/2}{4x^3}$$

$$= \lim_{x\to 0} \frac{-\cos(x) + \sin(x)(-\sin(x)) + \cos(x)\cos(x)}{12x^2}$$

$$= \lim_{x\to 0} \frac{-\cos(x) + \cos^2(x) - \sin^2(x)}{12x^2} = \lim_{x\to 0} \frac{\sin(x) - 2\cos(x)\sin(x) - 2\sin(x)\cos(x)}{24x}$$

$$= \lim_{x\to 0} \frac{\sin(x) - 4\sin(x)\cos(x)}{24x}$$

$$= \lim_{x\to 0} \frac{\cos(x) - 4\sin(x)(-\sin(x)) - 4(\cos(x))\cos(x)}{24} = \frac{1-4}{24} = \frac{-1}{8}.$$

Application

Optimization

21. Find the area of the smallest triangle formed by the x axis, y axis, and a line that goes through the point $(1,1)$.

Solution: Suppose that the line goes through the point $(0, y_0)$. Then, the slope of the line is $\frac{1-y_0}{1}$ and is described by the line $y - y_0 = \frac{1-y_0}{1}x$. The x intercept is when $y = 0$ or when $x = \frac{y_0}{y_0-1}$. Thus, the area of the triangle is

$$A(y_0) = \frac{1}{2} \cdot y_0 \cdot \frac{y_0}{y_0 - 1} = \frac{y_0^2}{2y_0 - 2}.$$

Setting the derivative equal to zero gives $A'(y) = \frac{y^2-2y(y-1)}{2(y-1)^2} = 0$ so $y^2 - 2y^2 + 2y = y(2-y) = 0$. The two solutions are $y = 0$ and $y = 2$. The second derivative is $\frac{-1}{(y-1)^3}$ and so $y_0 = 0$ is a local minimum and $y_0 = 2$ is a local maximum. So the maximum area is achieved by the line that goes through $(1,1)$ and $(0,2)$ with area $\frac{2\cdot 2}{2} = 2$.

22. Find the largest rectangle that can be inscribed into a semicircle of radius 2 so that one side of the rectangle is part of the diameter of the semicircle.

Solution: Let the height of the rectangle be h. Then the other side of the rectangle must be $2\sqrt{4-h^2}$. So we want to maximize $2h\sqrt{4-h^2}$, which is the same as maximizing its square $4h^2(4-h^2)$. Setting the derivative equal to 0 gives $32h - 16h^3 = 0$ so $h = \sqrt{2}$. The area is $2\sqrt{2} \cdot \sqrt{2} = 4$.

23. Find the point on the curve $y = 1 - \sqrt{x}$ closest to $(1,1)$.

Solution: The equation of distance is

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{(x-1)^2 + (1-\sqrt{x}-1)^2} = \sqrt{x^2 - 2x + 1 + x}$$

$$= \sqrt{x^2 - x + 1}.$$

Thus, we want to find when $\sqrt{x^2 - x + 1}$ is minimized. In order to find the minimum, we can take the derivative and set it equal to 0. Doing so gives

$$\frac{2x-1}{2\sqrt{x^2-x+1}} = 0 \implies x = \frac{1}{2}.$$

So, the point is $(1/2, 1 - \sqrt{1/2}) = (1/2, 1 - 1/\sqrt{2})$.

24. A rectangle is inscribed under the curve $\sin x$ for $0 \leq x \leq \pi$. This rectangle has two vertices on the curve and one side on the x axis. What is the maximum possible area of such a rectangle.

Solution: If one vertex on the x axis is at $(x_0, 0)$, then the other is at a point x' such that $\sin(x_0) = \sin(x')$. You can show that this point is $x' = \pi - x_0$. Thus, the vertices of this rectangle are $(x_0, 0), (x_0, \sin(x_0)), (\pi - x_0, 0), (\pi - x_0, \sin(x_0))$. Thus, the area of the rectangle is $(\pi - x_0 - x_0)\sin(x_0) = (\pi - 2x_0)\sin(x_0)$. Taking the derivative with respect to x_0 gives $(\pi - 2x_0)\cos(x_0) + (-2)\sin(x_0) = 0$ or $(\pi - 2x_0)\cos(x_0) = 2\sin(x_0)$. Solving gives $x_0 \approx 0.7105$ so the total area is $(\pi - 2(0.7105))\sin(0.7105) =\approx 1.122$.

25. What is the point on $y = e^x$ closest to $(1, 0)$?

Solution: We want to minimize the distance, which is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - 1)^2 + (e^x - 0)^2} = \sqrt{(x - 1)^2 + e^{2x}}.$$

We take the derivative and set it equal to 0 to get that

$$\frac{2(x - 1) + 2e^{2x}}{2\sqrt{(x - 1)^2 + e^{2x}}} = 0 \implies 2(x - 1) + 2e^{2x} = 0.$$

The solution is $x = 0$ since $2(-1) + 2(1) = 0$. Thus, the closest point is $(0, e^0) = (0, 1)$.

Related Rates

26. A ball of light is bobbing up and down and whose position is given at a time t by $4 + \sin(2t)$. A man who is $2m$ tall is standing $10m$ away. How fast is the length of his shadow changing when $t = 0$?

Solution: If the ball is at height d, then drawing a picture tells us that the height of the shadow satisfies the relation that $\frac{h}{2} = \frac{h+10}{d}$ so $dh = 2h + 20$. Taking the derivative of both sides gives $d'h + dh' = 2h'$. We have that $d' = 2\cos(2t)$ and when $t = 0$, we have that $d = 4 + \sin(0) = 4$ and $d' = 2\cos(0) = 2$. Solving for h gives $h = 10$ so

$$10 \cdot 2 + 4h' = 2h' \implies h' = -10.$$

So the shadow is shortening at $10m/s$.

27. A conical cup that is $6cm$ wide at the top and $5cm$ tall is filled with water is punctured at the bottom and water is coming out at a rate of $10^{-6}\pi m^3/s$. Initially, the cup is filled How fast is the height of the water changing when the height is $3cm$?

Solution: If the height of the water is h, then the radius of the cone formed by the water would be $3/5h$ and so the volume of the water cone is $V = \pi/3(3/5h)^2 \cdot h = \dfrac{3\pi h^3}{25}$. Taking the derivative of both sides gives

$$V' = \frac{9\pi h^2 h'}{25}$$

and plugging in $-10^{-6}\pi$ for V' and $3 \cdot 10^{-2}$ for h gives

$$-10^{-6}\pi = \frac{9\pi 9 \cdot 10^{-4} h'}{25} \implies h' = \frac{1}{81 \cdot 4} = \frac{1}{324} cm/s.$$

28. Sand is being dumped in a conical pile whose radius and height always remain the same. If the sand is being dumped in at a rate of $2\pi m^3/hr$, how fast is the height of the sand changing when the pile is $5cm$ tall?

Solution: Let the height of the pile be h. Then the radius of the pile is $r = h$ and the volume of the pile is $V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{3}$. Taking the derivative gives $V' = \pi h^2 h'$. Now we plug in 2π for V' and $5 \cdot 10^{-2}$ for h to get $h' = 800 m/hr = \frac{800}{3600} m/s = \frac{2}{9} m/s$.

29. A kite is flying at a current altitude of $100m$. The kite slowly flies further and further away as the string length increases at a rate of $2cm/s$. Assuming the altitude does not change, how fast horizontally is the kite moving when the angle the string forms with the ground is $\pi/4$?

Solution: Let ℓ be the length of the rope, and d how far horizontally the kite is flying. Then $\ell^2 = 100^2 + d^2$. Taking the derivative gives $2\ell\ell' = 2dd'$. When the angle the string forms with the ground is $\pi/4$, we calculate that $\ell = 100\sqrt{2}$ and $d = 100$ so $d' = \frac{100\sqrt{2} \cdot 2 \cdot 10^{-2}}{100} = 2\sqrt{2} \cdot 10^{-2} m/s$ or $2\sqrt{2} cm/s$.

30. A ladder $13m$ tall is lying against a wall. The bottom of the ladder is pulled out at a rate of $10cm/s$. How fast is the area of the triangle formed by the ladder, wall, and floor changing when the bottom of the ladder is $5m$ away from the wall?

Solution: Let d be how far the bottom of the ladder is away from wall. Then the area of the triangle formed is $\frac{1}{2} \cdot d \cdot \sqrt{169 - d^2} = A$. Squaring both sides gives

$4A^2 = d^2(169 - d^2)$. Now we can take the derivative to get that $8AA' = 2dd'(169 - d^2) + d^2(-2dd')$. When $d = 5$, the area is $\frac{1}{2} \cdot 5 \cdot 12 = 30$ and so

$$8 \cdot 30 \cdot A' = 2 \cdot 5 \cdot d'(144) + 25(-10d') \implies 240A' = 1190d'.$$

Since $d' = 10^{-1} m/s$, we have that $A' = \frac{119}{240} m^2/s$.

Taylor Series

31. Use the third order approximation to find $\sin(0.5)$.

Solution: We expand around 0 since $\sin 0 = 0$. We find that

$$\sin x \approx x - \frac{x^3}{6},$$

and so $\sin(0.5) \approx 0.5 - 0.5^3/6 \approx 0.4792$.

32. Approximate $\sqrt{99}$ using a quadratic regression.

Solution: We expand $f(x) = \sqrt{x}$ around $x = 100$ to get that

$$\sqrt{x} \approx f(100) + f'(100)(x - 100) + \frac{f''(100)}{2}(x - 100)^2 = 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}.$$

Thus, we have that

$$\sqrt{99} \approx 10 + \frac{-1}{20} - \frac{(-1)^2}{8000} = 10 - \frac{1}{20} - \frac{1}{8000}.$$

33. Use the second order approximation to find $\ln 1.01$.

Solution: We know that $\ln 1 = 0$. So we can expand out at $x = 1$ to get

$$\ln x \approx 0 + (x - 1) - \frac{(x - 1)^2}{2}.$$

Thus, we have that $\ln 1.01 \approx (0.01) - \frac{0.01^2}{2} = 0.00995$.

34. Use the second order approximation to $\sqrt[3]{8.1}$.

Solution: A close cube that we know is $2^3 = 8$. So we calculate the second order Taylor series expanded at $x = 8$ to get

$$\sqrt[3]{x} \approx 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}.$$

So plugging in 8.1 gives

$$\sqrt[3]{8.1} \approx 2 + \frac{.1}{12} - \frac{.1}{288} \approx 2.008.$$

35. Use the quintic order approximation to find e.

Solution: We know that $e^0 = 1$ so we can expand e^x around $x = 0$. Doing so gives

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$

Thus, we have that $e^1 \approx 1 + 1 + \frac{1}{2} = \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.717$.

Newton's Method

36. Use Newton's method once to approximate $\sqrt[3]{8.1}$.

Solution: We want to find the root of the equation $x^3 - 8.1$. Newton's method tells us to recursively apply the equation

$$x' = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 8.1}{3x^2}.$$

Our first guess is $x = 2$ and the next step is

$$2 - \frac{8 - 8.1}{3 \cdot 2^2} = 2 + \frac{1}{120} = 2.008.$$

37. Approximate $\sqrt{99}$ using Newton's method once.

Solution: We want to find the root of the equation $x^2 - 99$. Newton's method tells us to recursively apply the equation

$$x' = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 99}{2x}.$$

Our first guess is $x = 10$ and the next step is

$$10 - \frac{100 - 99}{20} = 10 - \frac{1}{20} = 9.95.$$

38. Find the critical points of $g(x) = \sin(x) - x^2$

Solution: We want to find when the derivative is 0 or when $f(x) = \cos(x) - 2x = 0$. Taking the derivative again, we find that it is $-\sin(x) - 2 < 0$ for all x. So this function is always decreasing and has a unique root. We plug in $x = 0$ to start, then calculate

$$x' = x - \frac{f(x)}{f'(x)} = -\frac{1}{-2} = \frac{1}{2}.$$

The real solution is ≈ 0.45.

39. Find the unique solution to $(\pi - 2x)\cos(x) = 2\sin(x)$ on the interval $[0, 1]$ using Newton's method with an initial guess of $x = \frac{\pi}{4}$.

Solution: We want to find the root of $f(x) = (\pi - 2x)\cos x - 2\sin x = 0$. Taking the derivative, we have that $f'(x) = -(\pi - 2x)\sin x - 4\cos x$. Now using Newton's method gets that

$$x' = x - \frac{f(x)}{f'(x)} = \frac{\pi}{4} - \frac{(\pi/2)(\sqrt{2}/2) - 2(\sqrt{2}/2)}{-(\pi/2)(\sqrt{2}/2) - 4(\sqrt{2}/2)} = \frac{\pi}{4} - \frac{\pi\sqrt{2} - 4\sqrt{2}}{-\pi\sqrt{2} - 8\sqrt{2}} \approx\approx 0.7084.$$

40. Find when $\cos(x) = x$ using Newton's method and an initial guess of $x = \frac{\pi}{6}$.

Solution: We want to find the root of $f(x) = \cos(x) - x = 0$. We calculate that $f'(x) = -\sin(x) - 1$. Newton's method gives us that

$$x' = x - \frac{f(x)}{f'(x)} = \frac{\pi}{6} - \frac{\sqrt{3}/2 - \pi/6}{-1/2 - 1} = \frac{\pi}{6} + \sqrt{3}3 - \frac{\pi}{9} = \frac{\pi}{18} + \sqrt{3}3 \approx 0.7519.$$

Functions

Domain/Range

41. Find the domain of $y = \sqrt{9 - (2x+3)^2}$.

Solution: The domain of $f(x) + \sqrt{9 - x^2}$ is $[-3, 3]$. So we have that $y = f(2x + 3)$ and to find the domain of y, we apply the linear transformations to f. So, first we subtract by 3 to get $[-6, 0]$ and then we divide by 2 to get $[-3, 0]$ as the domain for y. The reason we do it in reverse is that you should remember the general mantra that everything you expect would happen, the opposite happens for x.

42. Find the domain of $y = \frac{1}{\sqrt{3-x}}$.

Solution: First, the square root must be defined and so $3 - x \geq 0$ or $x \leq 3$. Then, we also need that the denominator cannot be 0 so $3 - x \neq 0$ or $x \neq 3$. Therefore, the domain is $\{x : x \leq 3 \wedge x \neq 3\} = \{x : x < 3\} = (-\infty, 3)$.

43. Find the domain and range of $2 - \arccos(3x + 2)$.

Solution: For the domain, the domain of $\arccos(x)$ is $[-1, 1]$. The linear shift $3x+2$ shifts the domain left by 2 then divides by 3 to get $[-3, -1]$ and then $[-1, -1/3]$, which is the domain. For the range, the range of arccos is $[0, \pi]$ and so $2-\arccos(3x+2)$ first multiplies the domain by -1 to get $[-\pi, 0]$ and then adds 2 to get $[2 - \pi, 2]$ as the range.

44. Find the domain of $\frac{\ln(x+3)}{\sqrt{2-x}}$.

Solution: The numerator must exist and hence $x + 3 > 0$ or $x > -3$. The denominator must exist and hence $2 - x \geq 0$ or $x \leq 2$. Also, the denominator cannot equal 0, so $x \neq 2$. The domain is the intersection of all 3 requirements and hence the domain is

$$\{x : x > -3 \wedge x \leq 2 \wedge x \neq 2\} = \{x : -3 < x < 2\} = (-3, 2).$$

45. Find the domain of $\sqrt{\dfrac{3-x}{1-x}}$.

Solution: We need that $\frac{3-x}{1-x} \geq 0$. If $x > 1$, then $1-x < 0$ and hence the inequality changes direction so multiplying by $1-x$ gives $3-x \leq 0$ or $x \geq 3$. If $x < 1$, then $1-x > 0$ so the inequality doesn't change direction so $3-x \geq 0$ or $x \leq 3$. Thus, the domain is the union of the regions

$$\{x : x > 1 \wedge x \geq 3\} \cup \{x : x < 1 \wedge x \leq 3\} = (-\infty, 1) \cup [3, \infty).$$

Inverse Functions

46. Find the inverse of $f(x) = \frac{-2}{x} - 1$.

Solution: We want to solve for x in terms of y so

$$x(y+1) = -2 \implies x = \frac{-2}{y+1}$$

is the inverse.

47. Find the inverse of $\frac{4+\sqrt{3x}}{5}$.

Solution: Solving for x in terms of x gives $\sqrt{3x} = 5y - 4$ so $x = (5y-4)^2/3$.

48. Find the inverse to x^2 on $(-\infty, 0]$.

Solution: The inverse to x^2 is $\pm\sqrt{x}$ but since the domain of x^2 is $(-\infty, 0]$, the range of the inverse should be $(-\infty, 0]$ which means that we should take the negative sign so the inverse is $-\sqrt{x}$.

49. Find the inverse to e^{2x+3}.

Solution: The inverse has $\ln y = 2x + 3$ so solving for x gives $x = \frac{-3 + \ln y}{2}$.

50. Find the inverse to $-\sqrt{\ln x}$.

Solution: Solving for x in terms of y gives $x = e^{y^2}$. But, the range of x is $(-\infty, 0]$ and hence the domain of the inverse is $(-\infty, 0]$. So the inverse is e^{-y^2} on $(-\infty, 0]$.

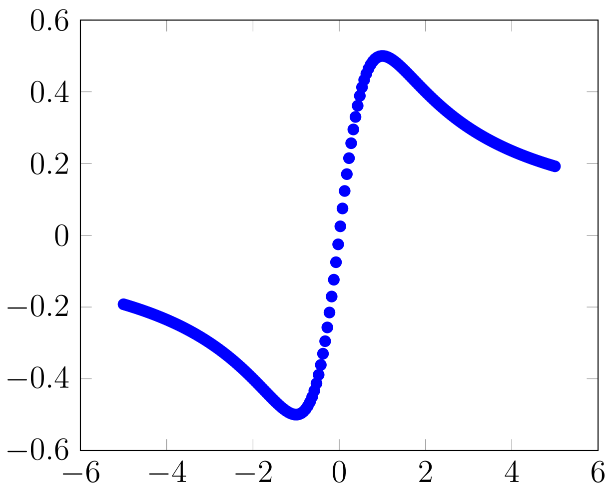
Graphing

51. Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$.

Solution: Take the derivative and second derivative to get $f'(x) = \dfrac{-(x^2-1)}{(x^2+1)^2}$ and $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x^2 - 1 = 0$ so $x = \pm 1$, and $f''(x) = 0$ gives $x(x^2-3) = 0$. So the points we need to put in our table are $x = 0, \pm 1, \pm\sqrt{3}$. We fill out the table the sign of f', f'' on these intervals to get

	$< -\sqrt{3}$	$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	$\sqrt{3} <$
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 0$. So there is a horizontal asymptote at $y = 0$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = -1$ and maximum at $x = 1$ by the second derivative test.

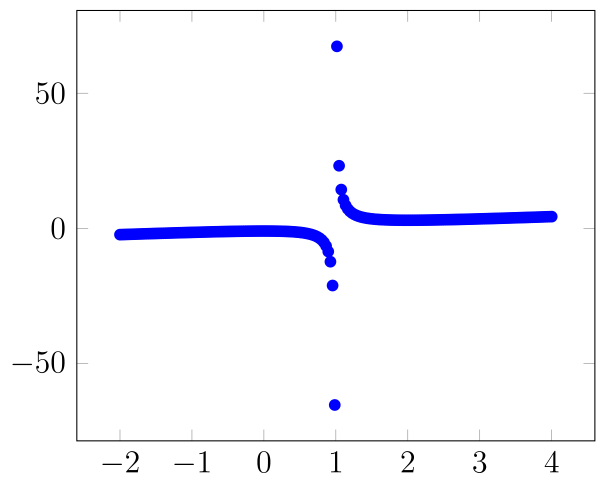


52. Sketch the graph of $f(x) = x + \dfrac{1}{x-1}$.

Solution: Take the derivative and second derivative to get $f'(x) = 1 - \frac{1}{(x-1)^2}$ and $f''(x) = \frac{2}{(x-1)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are not defined when $x = 1$ and solving $f'(x) = 0$ gives $(x-1)^2 = 1$ so $x = 0, 2$, and $f''(x) = 0$ has no solutions. So the points we need to put in our table are $x = 0, 1, 2$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, \infty)$
$f'(x)$	+	0	−	DNE	−	0	+
$f''(x)$	−	−	−	DNE	+	+	+

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$. Then since f is not defined at $x = 1$, we calculate the limits of f there with $\lim_{x\to1^-} f(x) = -\infty$, $\lim_{x\to1^+} f(x) = \infty$. So there is a vertical asymptote at $x = 1$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = 0$ and maximum at $x = 2$ by the second derivative test.

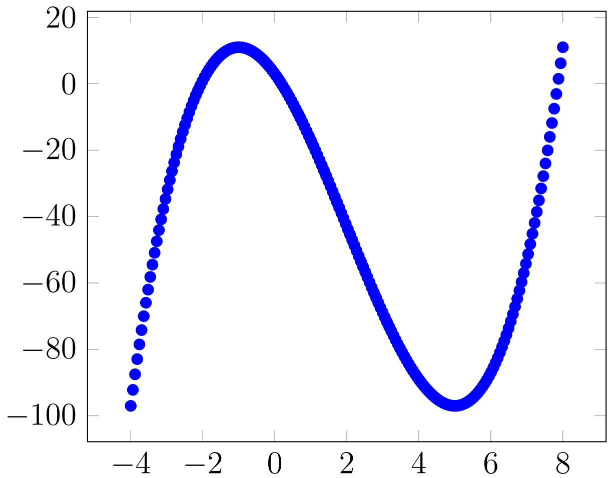


53. Sketch the graph of $f(x) = 3 - 15x - 6x^2 + x^3$.

Solution: Take the derivative and second derivative to get $f'(x) = -15 - 12x + 3x^2$ and $f''(x) = 6x - 12$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $x^2 - 4x - 5 = 0$ so $x = -1, 5$, and $f''(x) = 0$ gives $x = 2$. So the points we need to put in our table are $x = -1, 2, 5$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, -1)$	−1	$(-1, 2)$	2	$(2, 5)$	5	$(5, \infty)$
$f'(x)$	+	0	−	−	−	0	+
$f''(x)$	−	−	−	0	+	+	+

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = 5$ and maximum at $x = -1$ by the second derivative test.

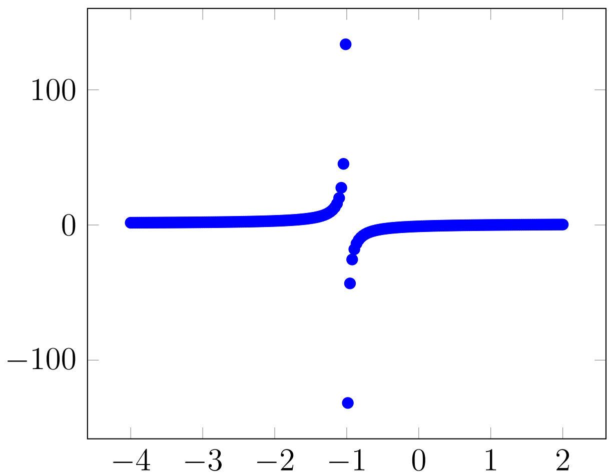


54. Sketch the graph of $f(x) = \dfrac{x-1}{x+1}$.

Solution: Take the derivative and second derivative to get $f'(x) = \dfrac{2}{(x+1)^2}$ and $f''(x) = \dfrac{-4}{(x+1)^3}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are not defined at $x = -1$ and solving $f'(x) = 0, f''(x) = 0$ has no solutions. So the points we need to put in our table is just $x = -1$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty,-1)$	-1	$(-1,\infty)$
$f'(x)$	$+$	DNE	$+$
$f''(x)$	$+$	DNE	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 1$. So there is a horizontal asymptote at $y = 1$. Now we calculate what happens as $x \to -1$ and we have $\lim_{x\to-1^-} f(x) = \infty$, $\lim_{x\to-1^+} f(x) = -\infty$ so it has a vertical asymptote at $x = -1$. We can now use this to produce something similar to the following graph.

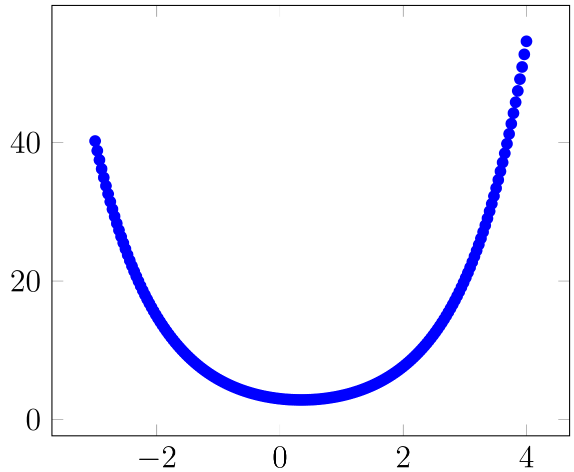


55. Sketch the graph of $f(x) = e^x + 2e^{-x}$.

Solution: Take the derivative and second derivative to get $f'(x) = e^x - 2e^{-x}$ and $f''(x) = e^x + 2e^{-x}$. We want to make a table and the values we care about are when $f'(x) = 0$, $f''(x) = 0$, and when they are not defined. They are always defined and solving $f'(x) = 0$ gives $e^{2x} = 2$ so $x = \frac{\ln 2}{2}$, and $f''(x) = 0$ has no solutions. So the point we need to put in is just $x = \frac{\ln 2}{2}$. We fill out the table the sign of f', f'' on these intervals to get

	$(-\infty, (\ln 2)/2)$	$(\ln 2)/2$	$((\ln 2)/2, \infty)$
$f'(x)$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$

Now we calculate the limits as $x \to \pm\infty$. We have $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$. We can now use this to produce something similar to the following graph noting that f will have a local minimum at $x = \frac{\ln 2}{2}$ by the second derivative test.


40
20
0
−2
0
2
4